OS-SE Waveguide

Marcel Batík

author@at-horns.eu

October 2020

1 Introduction

This text summarizes the newly developed OS-SE acoustic waveguide profile formula. This formula has already been used in the program *Ath*[1] developed by the author of this document, that also serves as a further reference.

The starting point for the developmnent was an oblate spheroidal (OS) waveguide as studied by Freehafer [1] and later independently by Geddes [2-4]. The profile curve of an OS waveguide is a hyperbola and by itself is infinite, without a natural termination required for a decent acoustic performance in any real-world situation. The most common way of terminating such a device would be to cut the profile at some point and, more or less smoothly, connect a segment of another curve suitable for this purpose. Preferably both curves should have the same slope and curvature at the junction point. One of the most suitable curves for this purpose is the Euler spiral, also known as Cornu spiral or clothoid. Although very effective, practical use of this curve can be somewhat difficult as it is not available in a simple closed analytic form.

After some experimantation with the clothoids it was noticed that very similar results can be obtained by adding a second function, representing a quadrant of a superellipse to the basic profile. The main advantage of this approach is that it provides a simple closed-form function for the whole new profile, allowing an easy manipulation in analytic form. Since the underlying profile and the termination is now smoothly blended together, a completely seamless transition is achieved as the whole profile curve is infinitely differentiable. It also makes the whole profile highly configurable by a simple and intuitive set of numerical parameters.

1 https://at-horns.eu/

2 Notation

In the following text a cartesian coordinate system with axes x, y, z is considered, with a waveguide throat centered at origin, facing in the direction of the positive z-axis.

Any point [x, y, z] on the surface of the waveguide can be also expressed in polar coordinates in the x-y plane of the respective cross section:

$$[x, y, z] = [r, \phi]\big|_z$$

r horn "radius", i.e the distance of the surface point from the z axis

ϕ angle around the horn's axis (<0, 2π>; zero angle corresponds to a direction of the x axis)

In the presented notation a general waveguide surface can be described as $r(z, \phi)$.

3 Generalized OS Waveguide

OS waveguide in its pure (infinite) form has the following profile formula (see Fig. 1):

$$r_{OS}(z) = \sqrt{r_0^2 + z^2 \tan^2(\alpha)} \quad (1)$$

r_0 throat radius

α nominal coverage angle (half the beamwidth)

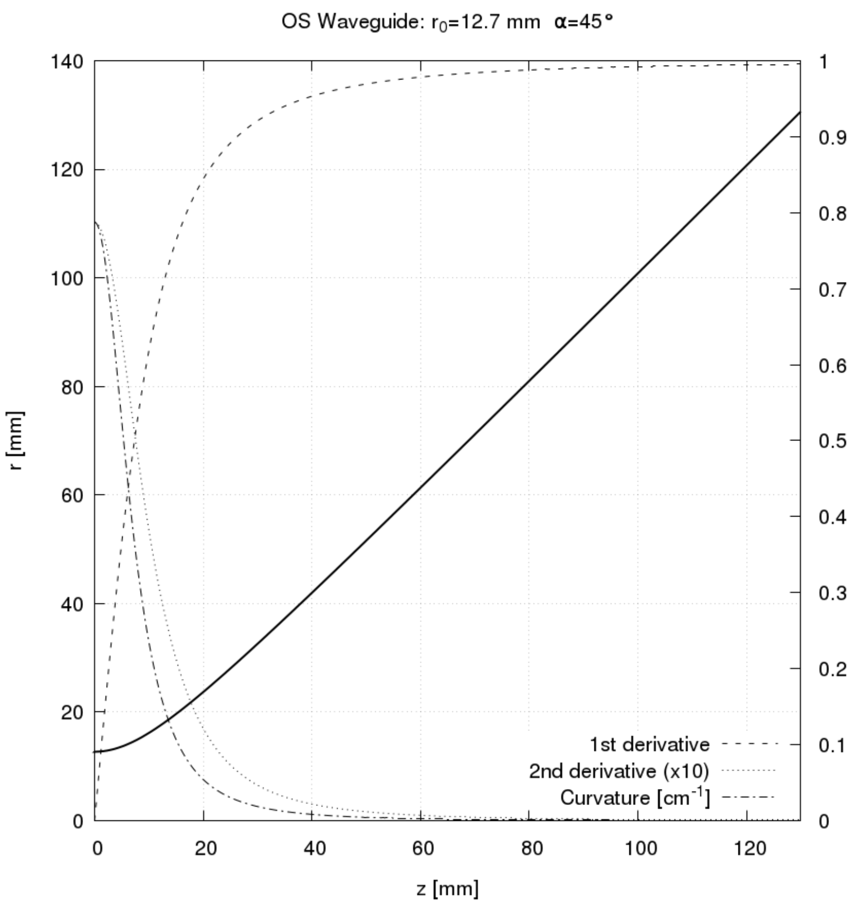

Figure 1: OS waveguide

In the basic form (1) the hyperbola that this formula represents is centered at origin and the throat opening angle is zero, matching a flat wavefront. In many practical situations a different throat opening angle is desirable. For this purpose the hyperbola must be shifted along the z-axis and scaled accordingly (see Fig. 2):

$$r_{OS}(z) = \sqrt{r_0^2 + 2 r_0 z \tan(\alpha_0) + z^2 \tan^2(\alpha)} \tag{2}$$

α_0 throat opening angle (half the included angle)

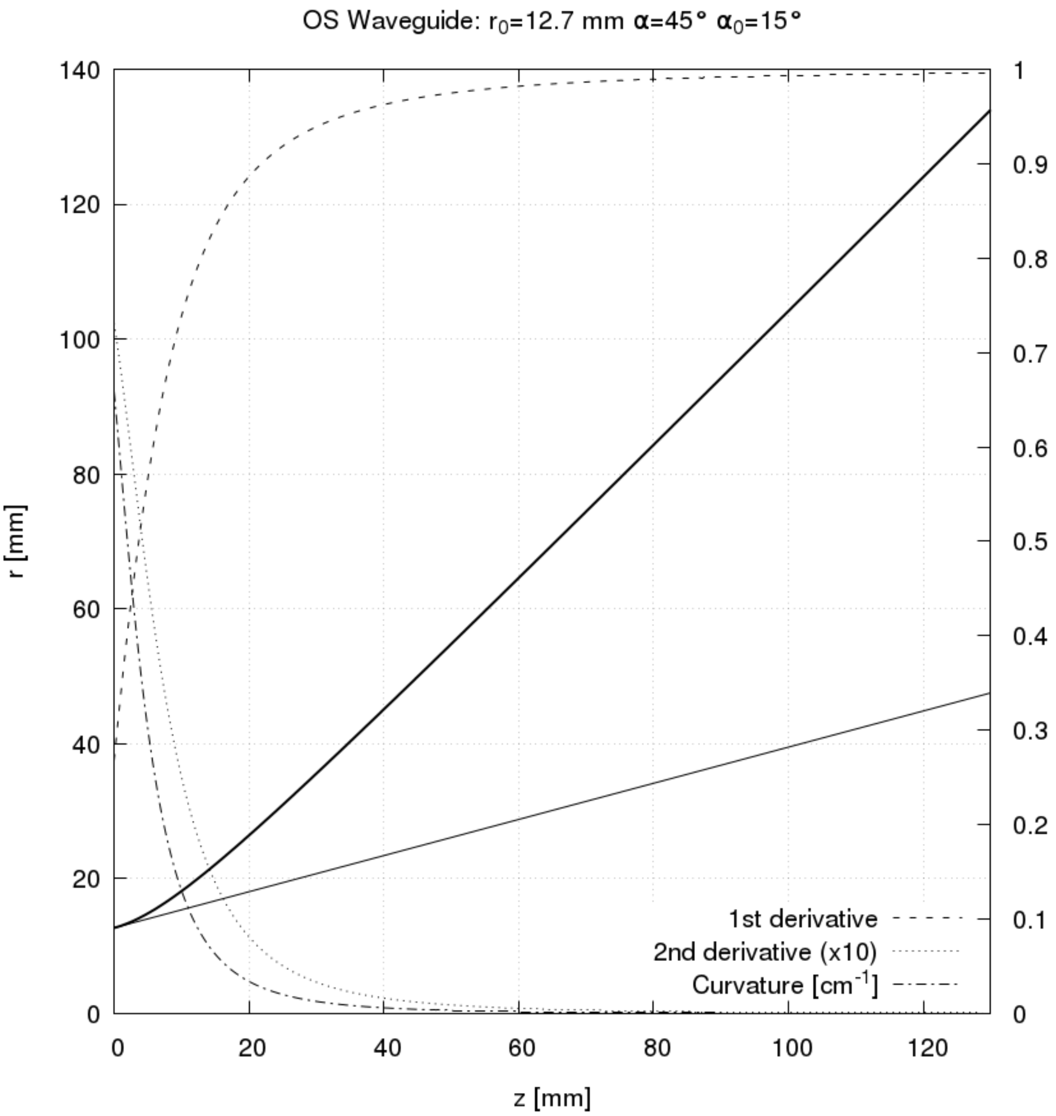

Figure 2: OS waveguide (throat opening angle = 30°)

Another possible generalization abandones the OS coordinate system in a strict sense but offers additional level of flexibility for the waveguide design:

$$r_{GOS}(z) \ = \ \sqrt{k^2 r_0^2 + 2kr_0 z \tan(\alpha_0) + z^2 \tan^2(\alpha)} \ + \ r_0(1-k) \qquad (3)$$

k throat expansion factor

The above formula basically enables to use an OS profile (still a hyperbola) that would otherwise correspond to a different throat radius – larger for k > 1 and smaller for k < 1. The formula still represents a pure OS profile for k = 1.

Note that for k = 0 the profile collapses into a simple conical waveguide, i.e. into a straight line:

$$r_{GOS}(z)\big|_{k=0} \ = \ r_0 \ + \ z \tan(\alpha)$$

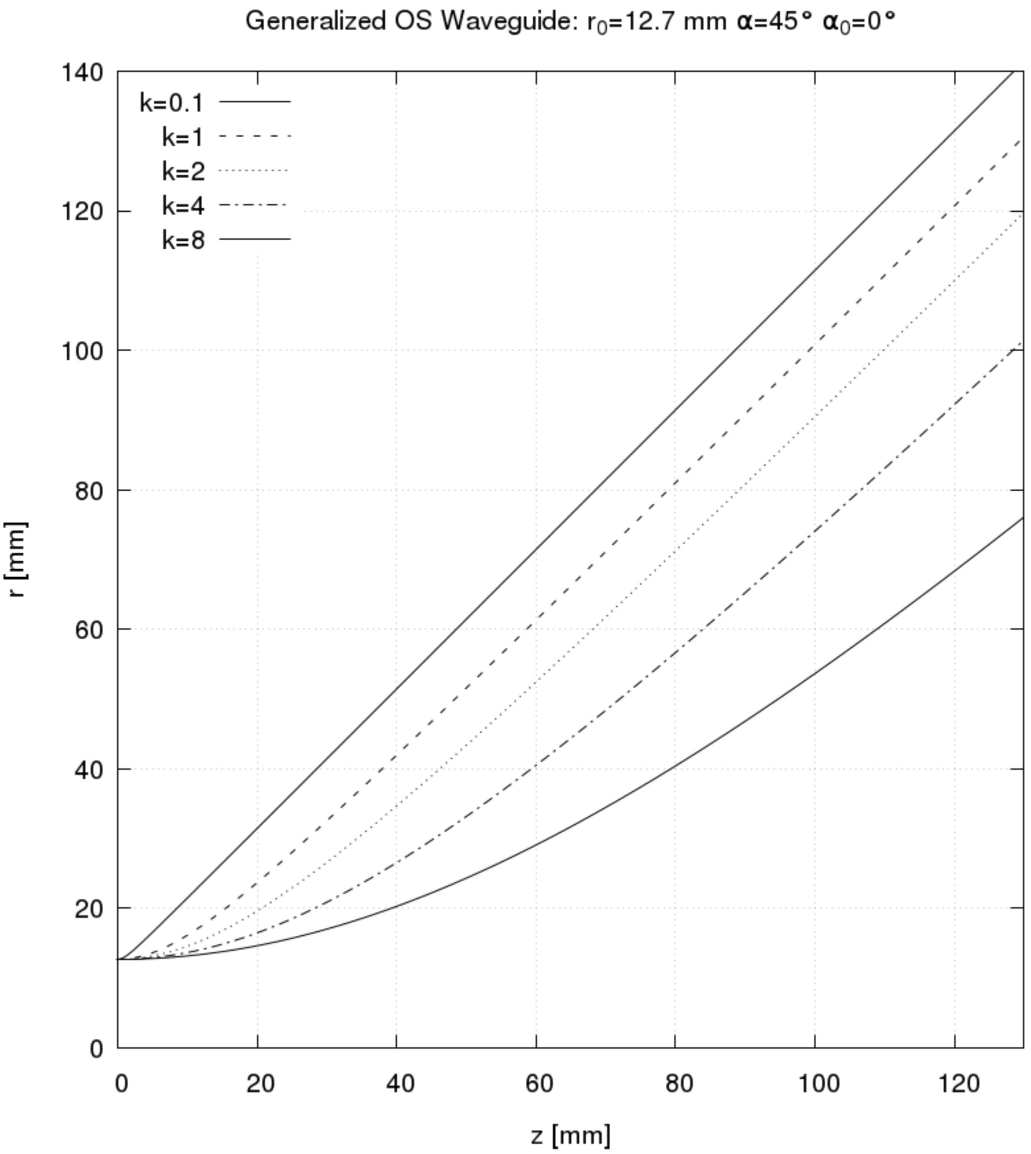

Figure 3: Generalized OS waveguide

4 The Termination

The generalized OS profile formula (3) presented in the previous section still describes an infinite device. To terminate the profile with a mouth flare in a finite length, another term can be added, giving a general formula of the profile:

$$r(z) = r_{GOS}(z) + r_{TERM}(z) \quad , 0 \le z \le L$$

L length of the waveguide

After some experimentation it was found out that an arc of a superelliptical[2] (SE) quadrant works very well for this purpose, closely resembling a termination with a segment of a clothoid (see Fig. 4).

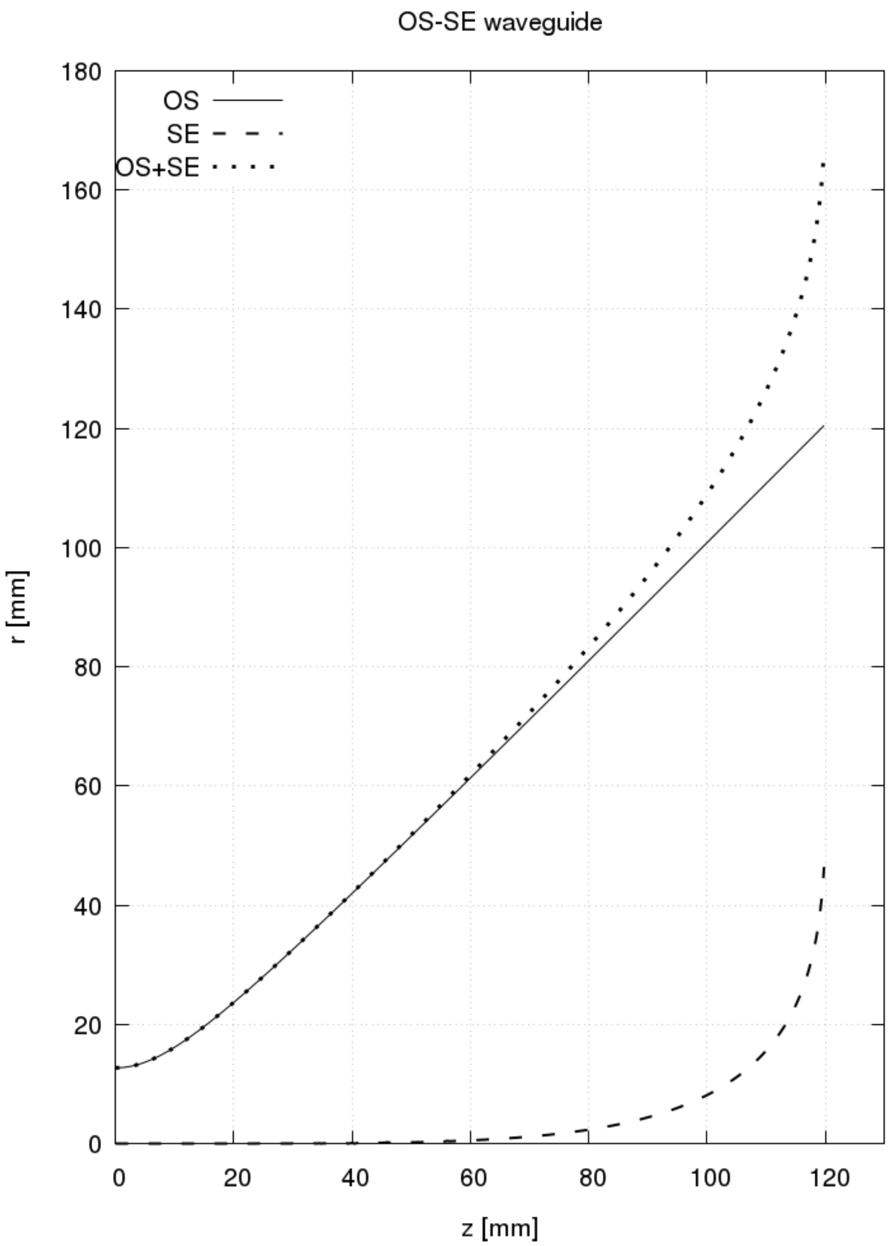

Figure 4: Terminated OS waveguide (length = 120 mm)

2 https://en.wikipedia.org/wiki/Superellipse

Superellipse with *a*, *b* being a semi-major and semi-minor axis is a set of points satisfying the following equation (x, y ≥ 0; n ≥ 2):

$$\left(\frac{z}{a}\right)^n + \left(\frac{r}{b}\right)^n = 1$$

To get an expression suitable as a function r(z) it takes the following form:

$$r_{SE}(z) = b - b\left(1-\frac{z^n}{a^n}\right)^{\frac{1}{n}} = b\left[1 - \left(1 - \frac{z^n}{a^n}\right)^{\frac{1}{n}}\right]$$

With the semi-major axis equal to the horn length (*L*) and the semi-minor axis to its arbitrary multiple (*sL*) it becomes:

$$r_{SE}(z) = sL\left[1 - \left(1 - \frac{z^n}{L^n}\right)^{\frac{1}{n}}\right]$$

s superellipse aspect ratio (amount of termination flare)

The parameter 's' basically determines the amount of flare applied to the profile and the resulting mouth radius. Value of zero means no flaring, i.e. keeping the profile intact.

The above formula would be already fairly usable but one minor yet helpful modification is still possible - typically it will be more convenient to not include the whole quadrant of an arc, as the very last portion would add nothing more than a short segment of more or less straight line, increasing the overall size of the device without much benefit. For this purpose, to use a little bit less than the whole quadrant, a parameter *q* (truncation coefficient) is introduced. This gives the final additive flare term:

$$r_{TERM}(z) = \frac{sL}{q}\left[1 - \left(1 - \left(\frac{qz}{L}\right)^n\right)^{\frac{1}{n}}\right] \qquad (4)$$

q truncation coefficient (typical value: 0.99 - 1.00)

The so far not examinated parameter *n* (exponent of the superellipse) mainly affects how far from the throat the terminating term becomes effective and how gradual the termination is. The higher the value the more of the underlying profile remains preserved in the final shape and the more rapid is the termination and the change of curvature near the end.

To sum up, a smoothly terminated generalized OS waveguide is defined by the following formula (Fig. 5):

$$r_{OSSE}(z) \;=\; \sqrt{k^2 r_0^2 + 2kr_0 z\tan(\alpha_0) + z^2\tan^2(\alpha)} \;+\; r_0(1-k) \;+\; \frac{sL}{q}\left[1-\left(1-\left(\frac{qz}{L}\right)^n\right)^{\frac{1}{n}}\right] \qquad (5)$$

The figures on the following pages show effects of some of the parameters on resulting OS-SE profile.

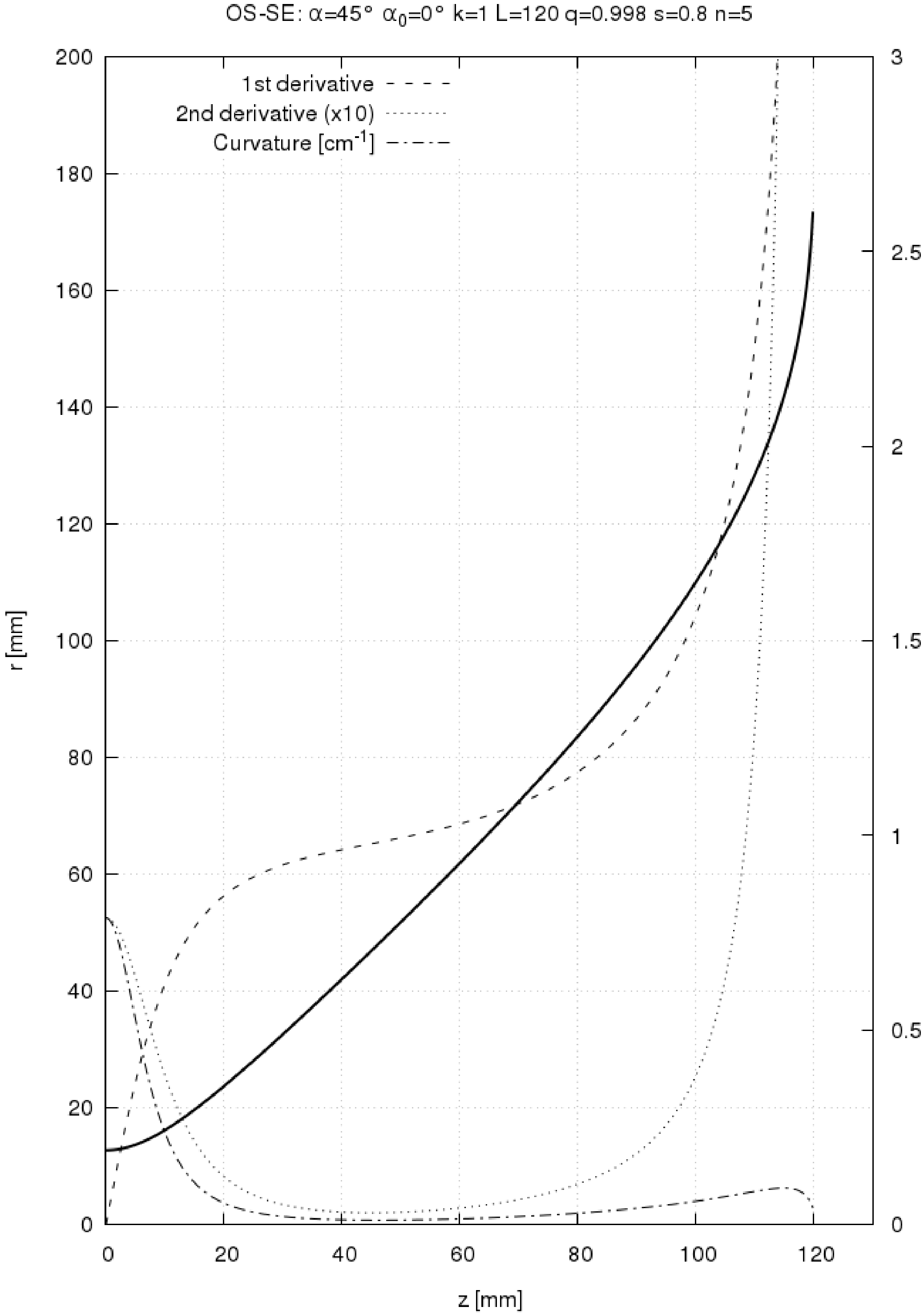

Figure 5: OS-SE waveguide

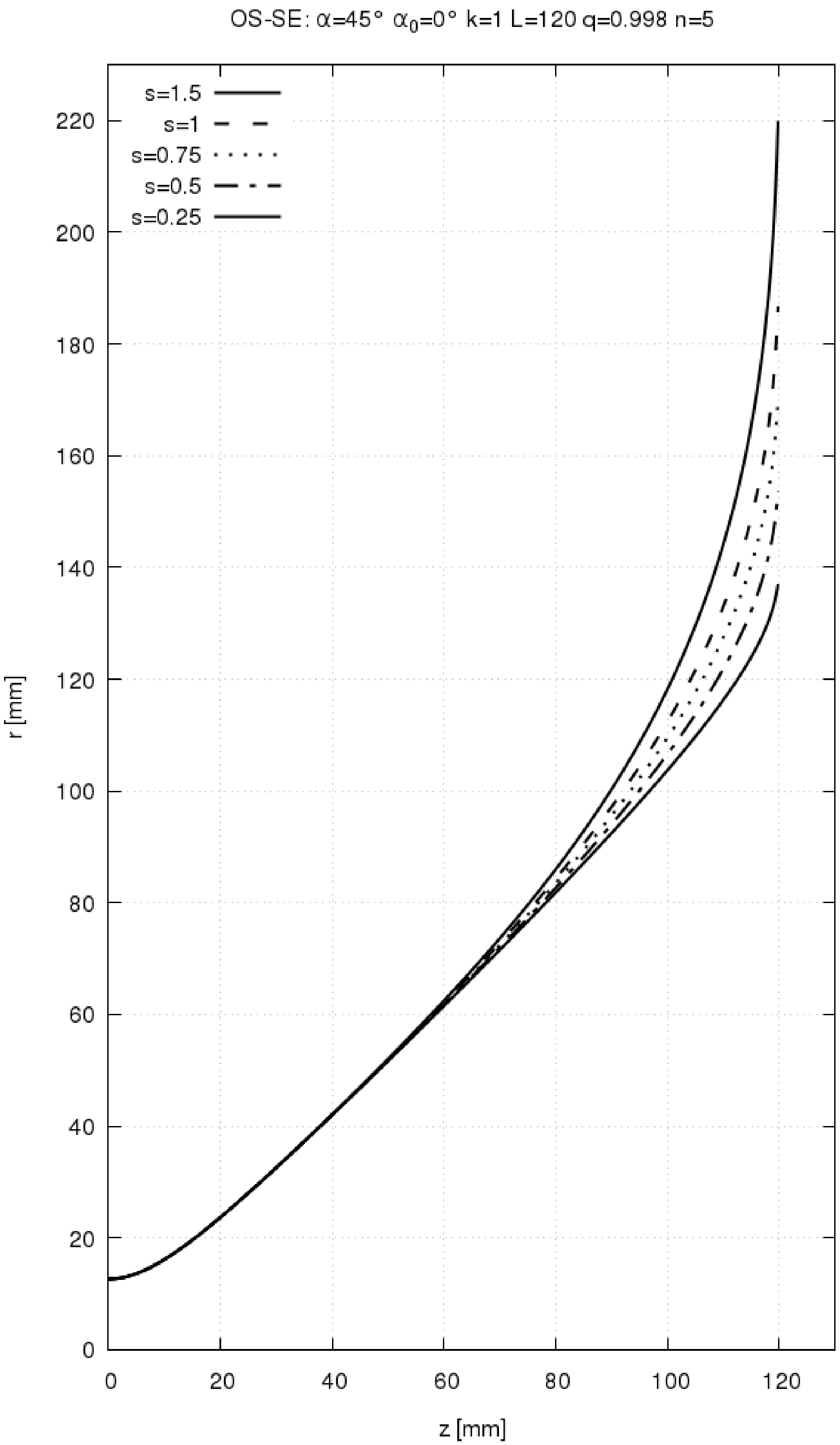

Figure 6: OS-SE waveguide (effect of "s")

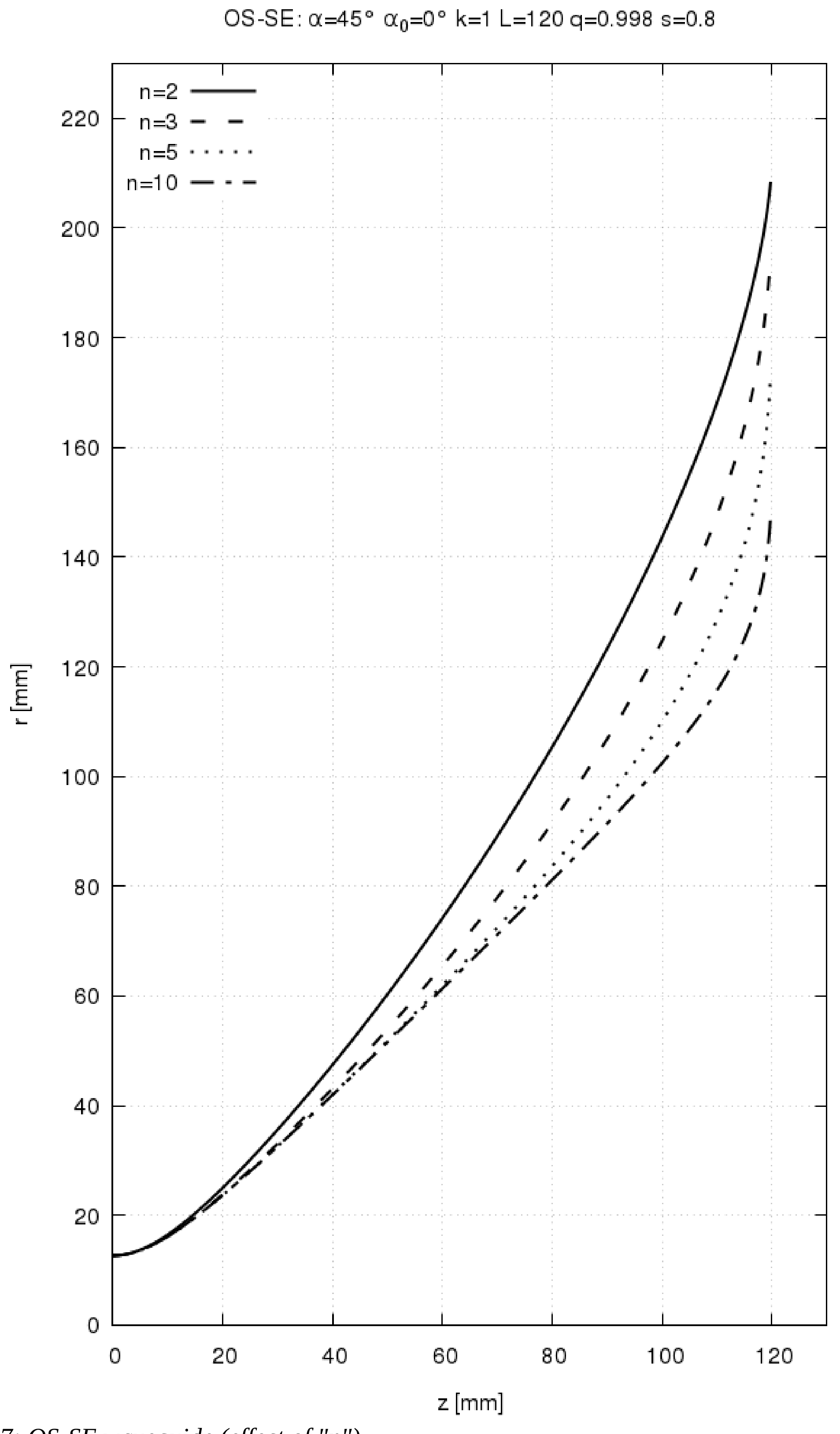

Figure 7: OS-SE waveguide (effect of "n")

5 Morphing

Normally the mouth outline shape $r(L,\phi)$ is determined by the profile(s) used (note that the OS-SE profile parameters may vary around the device, i.e. not necessarily being axisymmetric). There is however a possibility to transform this shape into another one with a different mouth outline. This feature, implemented in the Ath tool, is called morphing and the algorithm takes an existing surface of a waveguide and gradually transforms it from the thorat to the mouth so that it reaches the desired mouth outline exactly at the full length.

In general, an arbitrary target mouth outline is given by a function $r_M(\phi)$.

There doesn't have to be an analytic expression for the target mouth outline - all that's needed is that it can be determined for any value of ϕ .

The morphing algorithm, which calculates a new horn radius r_m at each profile point, uses the following transformation:

1) for $z < z_f$ the existing profile is preserved intact:

$$r_m(z,\phi) = r(z,\phi)$$

2) for $z \geq z_f$ the profile is gradually changed towards the mouth:

$$r_m(z,\phi) = r(z,\phi) + \left(\frac{z-z_f}{L-z_f}\right)^{\gamma}\left(r_M(\phi)-r(L,\phi)\right)$$

z_f	fixed part of the original shape that won't be modified be the transformation
$r_M(\phi)$	radius of the target mouth outline
γ	morph rate, $\gamma \geq 1$

How gradual the transition from the original to the target shape should be is determined by the parameter γ. For $\gamma = 1$, there's an abrupt slope change at $z=z_f$. For higher values of γ the transition will be more gradual and slower.

Obviously, the transformed profile has the following properties:

$$r_m(0,\phi)=r(0,\phi)=r_0$$

$$r_m(L,\phi)=r_M(\phi)$$

6 Conclusion

The OS-SE waveguide formula was presented. In this analytical form it is suitable for a low-diffraction waveguide terminated in a flat baffle. In a case of free standing waveguide, this profile should be terminated further to eliminate diffration and reflection at the mouth. One possible solution is to extend the profile by connecting a segment of a clothoid, "rolling" the profile naturally further back. Unfortunately, there is no simple analytical form available (at least not known to the author) and such termination must be done numerically.

Acoustic properties of OS-SE waveguides, either placed in infinite baffles or free standing (terminated by a mouth rollback), can be analyzed by means of the program *Ath*, developed by the author and freely avalable at http://at-horns.eu.

7 References

[1] Freehafer, J.E. (1940), The Acoustical Impedance of an Infinite Hyperbolic Horn, JAES 11: 467-76.

[2] Geddes, E. R. (1989), Acoustic Waveguide Theory, Journal Audio Eng. Soc. 37-7.

[3] Geddes, E. R. (1993), Acoustic Waveguide Theory Revisited, Journal Audio Eng. Soc. 41-6.

[4] Geddes, E. R. (2002), Audio Transducers, GedLee LLC.